Moving grids for hyperbolic problems

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Objectives (1/4)

Let us consider the one-dimensional viscous Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad x \in (0,1), \qquad t \in (0,T),$$

subject to the initial condition

$$u(x,0) = u_0(x).$$

The look-ahead strategy:

- Assume that the data at $t = t^n$ are given and exact.
- Derive exact and approximate error functionals at $t = t^{n+1}$.
- Build an adapted grid minimizing the approximate error functional.
- Interpolate data to the adapted grid and perform one time step.

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Objectives (2/4)

Let $t = t^n$. We consider a grid

$$0 = x_0^n < x_1^n < \ldots < x_{M+1}^n = 1$$

and define

$$x_{i+1/2}^n = (x_{i+1}^n + x_i^n)/2, \qquad h_{i+1/2}^n = x_{i+1}^n - x_i^n.$$

Let

$$\bar{u}_{i+1/2}^0 = \frac{1}{h_{i+1/2}} \int_{x_i^0}^{x_{i+1}^0} u_0(x) \mathrm{d}x.$$

Objectives (3/4)

Let us analyze an error functional associated with the donor scheme:

$$\begin{aligned}\bar{u}^{n+1}_{i+1/2} &= \mathcal{L}^n_{i+1/2}(\bar{u}^n) \\ &= \bar{u}^n_{i+1/2} - \frac{\Delta t^n}{h_{i+1/2}}(f^n_{i+1} - f^n_i) + \frac{\varepsilon \Delta t^n}{h^n_{i+1/2}}\left(\left[\frac{\delta u^n}{\delta x}\right]_{i+1} - \left[\frac{\delta u^n}{\delta x}\right]_i\right)\end{aligned}$$

where f^n_i denotes the flux at point x^n_i,

$$f^n_i = \frac{1}{2}\begin{cases}(\bar{u}^n_{i+1/2})^2 & \text{if} \quad \bar{u}_{i+1/2} + \bar{u}_{i-1/2} \geq 0, \\ (\bar{u}^n_{i-1/2})^2 & \text{otherwise,}\end{cases}$$

and

$$\min_i \Delta t^n \left(\frac{\bar{u}^n_{i+1/2}}{h^n_{i+1/2}} + \frac{2\varepsilon}{(h^n_{i+1/2})^2}\right) < 1.$$

Objectives (4/4)

Consider the following minimization problem:

$$\min_{x_1^n,\ldots,x_M^n} F_{ex}^n(\{x_i^n\})$$

where

$$F_{ex}^n = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x,\, t^{n+1}) - \bar{u}_{i+1/2}^{n+1}|^2 \mathrm{d}x = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x,\, t^{n+1}) - \mathcal{L}_{i+1/2}^n(\bar{u}^n)|^2 \mathrm{d}x$$

The stability condition and Taylor expansion give

$$\int_{x_i^n}^{x_{i+1}^n} \left(u(x,\, t^{n+1}) - \mathcal{L}_{i+1/2}^n(\{\bar{u}^n\}) \right)^2 \mathrm{d}x =$$

$$\left(\left.\frac{\partial u}{\partial x}\right|_{x_{i+1/2}^n} \right)^2 \left[\frac{(h_{i+1/2}^n)^3}{12} + \frac{(\Delta t^n\, u_{i+1/2}^n)^2}{4} \frac{(h_{i+1/2}^n - h_{i-1/2}^n)^2}{h_{i+1/2}^n} \right] + O(h_{i+1/2}^n)^4.$$

Objectives (4/4)

Consider the following minimization problem:

$$\min_{x_1^n,\ldots,x_M^n} F_{ex}^n(\{x_i^n\})$$

where

$$F_{ex}^n = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x,\, t^{n+1}) - \bar{u}_{i+1/2}^{n+1}|^2 \mathrm{d}x = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x,\, t^{n+1}) - \mathcal{L}_{i+1/2}^n(\bar{u}^n)|^2 \mathrm{d}x$$

The stability condition and Taylor expansion give

$$\int_{x_i^n}^{x_{i+1}^n} \Big(u(x,\, t^{n+1}) - \mathcal{L}_{i+1/2}^n(\{\bar{u}^n\})\Big)^2 \mathrm{d}x \approx \left(\left.\frac{\partial u}{\partial x}\right|_{x_{i+1/2}^n} \right)^2 \left[\frac{(h_{i+1/2}^n)^3}{12} \right].$$

Exact error functional (1/2)

Consider a grid

$$0 = x_0 < x_1 < \ldots < x_{M+1} = 1$$

and define

$$x_{i+1/2} = (x_{i+1} + x_i)/2, \qquad h_{i+1/2} = x_{i+1} - x_i.$$

Let $f^h(x)$ be a piecewise constant approximation of $f(x)$. Then, the minimum of the functional

$$\Phi(\{x_i\}, \{\bar{f}_{i+1/2}\}) = \int_0^1 \left(f(x) - f^h(x)\right)^2 \mathrm{d}x = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} \left(f(x) - \bar{f}_{i+1/2}\right)^2 \mathrm{d}x$$

is achieved when

$$\bar{f}_{i+1/2} = \frac{1}{h_{i+1/2}} \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x.$$

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Exact error functional (2/2)

Thus, the problem

$$\min_{x_1,\dots,x_M,\ \bar{f}_{1/2},\dots,\bar{f}_{M+1/2}} \Phi(\{x_i\},\{\bar{f}_{i+1/2}\})$$

is reduced to

$$\min_{x_1,\dots,x_M} F_{ex}(\{x_i\}), \qquad F_{ex}(\{x_i\}) = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} \left(f(x) - \bar{f}_{i+1/2}\right)^2 \mathrm{d}x.$$

The Taylor expansion with the Lagrange remainder gives

$$\min_{x_1,\dots,x_M} F_{ex}(\{x_i\}), \qquad F_{ex}(\{x_i\}) = \frac{1}{12}\sum_{i=0}^{M} \left(\left.\frac{\partial f}{\partial x}\right|_{x^*_{i+1/2}} \right)^2 h^3_{i+1/2}$$

where $x^*_{i+1/2}$ is a point from interval $(x_i,\ x_{i+1})$.

Minimization & equidistribution (1/4)

Lemma. Let $e_{i+1/2}(\cdot,\,\cdot)\colon \Re^2 \to \Re$ be a set of functions defined by

$$e_{i+1/2}(x_i,\,x_{i+1}) = \int_{x_i}^{x_{i+1}} g(x)\mathrm{d}x, \qquad 0 \le x_i \le x_{i+1} \le 1,$$

where $g(x) \ge 0$ is an arbitrary bounded function. Then

$$\min_{x_1,\ldots x_M} \sum_{i=0}^{M} e^p_{i+1/2}(x_i,\,x_{i+1}) = \frac{\mathcal{E}^p}{(M+1)^{p-1}}$$

where p is a positive integer and

$$\mathcal{E} = \sum_{i=0}^{M} e_{i+1/2}(x_i,\,x_{i+1}) = \int_0^1 g(x)\mathrm{d}x.$$

Moreover, the minimum is achieved when $e_{i+1/2}(x_i,\,x_{i+1}) = \mathcal{E}/(M+1)$.

Minimization & equidistribution (1/4)

Lemma. Let $e_{i+1/2}(\cdot,\,\cdot) : \Re^2 \to \Re$ be a set of functions defined by

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Moreover, the minimum is achieved when $e_{i+1/2}(x_i,\,x_{i+1}) = \mathcal{E}/(M+1)$.

Minimization & equidistribution (2/4)

We introduce additional notations:

$$\hat{\omega}_{i+1/2} = \left(\frac{1}{12} \left. \frac{\partial f}{\partial x} \right|_{x^*_{i+1/2}} \right)^{2/3} \qquad \text{and} \qquad \hat{e}_{i+1/2} = \hat{\omega}_{i+1/2} h_{i+1/2}.$$

Then, we can rewrite the functional F_{ex} as follows:

$$F_{ex}(\{x_i\}) = \frac{1}{12} \sum_{i=0}^{M} \left(\left. \frac{\partial f}{\partial x} \right|_{x^*_{i+1/2}} \right)^2 h^3_{i+1/2} = \sum_{i=0}^{M} \hat{e}^3_{i+1/2} = \sum_{i=0}^{M} \hat{\omega}^3_{i+1/2} h^3_{i+1/2}.$$

It is obvious that

$$\hat{e}_{i+1/2} \to \int_{x_i}^{x_{i+1}} \left| \frac{\partial f}{\partial x} \right|^{2/3} \mathrm{d}x \quad \text{and} \quad \sum_{i=0}^{M} \hat{e}_{i+1/2} \to \int_0^1 \left| \frac{\partial f}{\partial x} \right|^{2/3} \mathrm{d}x.$$

Minimization & equidistribution (2/4)

We introduce additional notations:

$$\hat{\omega}_{i+1/2} = \left(\frac{1}{12} \left. \frac{\partial f}{\partial x} \right|_{x^*_{i+1/2}} \right)^{2/3} \qquad \text{and} \qquad \hat{e}_{i+1/2} = \hat{\omega}_{i+1/2} h_{i+1/2}.$$

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In other words, taking $g(x) = |\partial f / \partial x|^{2/3}$, we get

$$\hat{e}_{i+1/2} \rightarrow \int_{x_i}^{x_{i+1}} g(x) \mathrm{d}x \qquad \text{and} \qquad \sum_{i=0}^{M} \hat{e}_{i+1/2} \rightarrow \int_0^1 g(x) \mathrm{d}x.$$

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Minimization & equidistribution (3/4)

The equidistribution principle,

$$\hat{e}_{i+1/2} = \hat{e}_{i-1/2}, \qquad i = 1, \ldots, M,$$

may be rewritten as follows:

$$\hat{\omega}_{i+1/2}(x_{i+1} - x_i) - \hat{\omega}_{i-1/2}(x_i - x_{i-1}) = 0.$$

It is a discretization of the non-linear elliptic equation

$$\frac{\partial}{\partial \xi}\left(\omega(x)\frac{\partial x}{\partial \xi}\right) = 0, \qquad x(0) = 0, \quad x(1) = 1,$$

on a uniform grid with the coefficient $\omega(x)$ given by

$$\omega(x) = \left|\frac{\partial f}{\partial x}\right|^{2/3}.$$

Minimization & equidistribution (4/4)

A discrete analog of the nonlinear elliptic equation can be directly derived from

$$\nabla F_{ex} = 0.$$

Recall that

$$F_{ex}(\{x_i\}) = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} \left(f(x) - \bar{f}_{i+1/2}\right)^2 \mathrm{d}x.$$

Then

$$\frac{\partial F_{ex}}{\partial x_i} = 2f(x_i) - \bar{f}_{i-1/2} - \bar{f}_{i+1/2} = 0.$$

The Taylor expansion at point x_i results in

$$\left.\frac{\partial f}{\partial x}\right|_{x_i} \left(h_{i+1/2} - h_{i-1/2}\right) + \frac{1}{3} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_i} \left(h^2_{i+1/2} + h^2_{i-1/2}\right) = 0.$$

Approximate error functional (1/7)

Let R_h be an interpolation operator from grid $\{x_i^0\}$ to grid $\{x_i\}$. Consider the following minimization problem:

$$\min_{x_1,\dots,x_M} \int_0^1 \left| f(x) - \left[R_h(f^{h,0}) \right](x) \right|^2 \mathrm{d}x.$$

We assume that

- R_h is exact for linear functions;
- R_h is conservative.

$$\int\limits_0^1 \left| f(x) - \left[R_h(f^{h,0}) \right](x) \right|^2 \mathrm{d}x = \sum_{i=0}^{M} \int\limits_{x_i}^{x_{i+1}} \left| f(x) - \bar{f}_{i+1/2} + O(h_{i+1/2}^2) \right|^2 \mathrm{d}x$$

$$= \sum_{i=0}^{M} \left[\frac{1}{12} \left(\left. \frac{\partial f}{\partial x} \right|_{x^*_{i+1/2}} \right)^2 h_{i+1/2}^3 + O(h_{i+1/2}^4) \right].$$

Approximate error functional (2/7)

Recall that

$$F_{ex} = \sum_{i=0}^{M} \hat{\omega}_{i+1/2}^3 h_{i+1/2}^3.$$

Since the precise computation of coefficients $\hat{\omega}_{i+1/2}$ is impossible, they are replaced by computable coefficients $\omega_{i+1/2}$ such that $\omega_{i+1/2} \approx \hat{\omega}_{i+1/2}$,

$$\omega_{i+1/2} = \frac{1}{h_{i+1/2}^3} \sum_{k=0}^{M} \int_{\check{x}_{ik}}^{\hat{x}_{ik}} \left| \bar{f}_{k+1/2}^0 + \left[\frac{\delta f^{h,0}}{\delta x} \right]_{k+1/2} (x - x_{k+1/2}) - [R_h(f^{0,h})](x) \right|^2 \mathrm{d}x$$

where $[\hat{x}_{ik},\, \check{x}_{ik}] = [x_i,\, x_{i+1}] \cap [x_k^0,\, x_{k+1}^0]$. This results in an approximate minimization problem:

$$\min_{x_1,\ldots,x_M} F_{ap}(\{x_i\}), \qquad F_{ap}(\{x_i\}) = \sum_{i=0}^{M} \omega_{i+1/2}^3 h_{i+1/2}^3.$$

Approximate error functional (3/7)

$$\frac{\partial}{\partial \xi}\left(\omega(x)\frac{\partial x}{\partial \xi}\right) = 0, \qquad x(0) = 0, \quad x(1) = 1,$$

Algorithm (equidistribution principle)

For $k = 1, \ldots, K_{max}$ do

1. For the given grid $\{x_i^k\}$ compute values $\omega_{i+1/2}^k$, $i = 0, \ldots, M$.
2. Perform one Gauss-Seidel sweep

$$\omega_{i+1/2}^k(x_{i+1}^k - x_i^{k+1}) - \omega_{i-1/2}^k(x_i^{k+1} - x_{i-1}^{k+1}) = 0, \qquad i = 1, \ldots, M.$$

3. Stop iterations if $\max_i |x_i^k - x_i^{k+1}| \leq TOL$ where TOL is the user given tolerance.

Approximate error functional (4/7)

$$\min_{x_1,\ldots,x_M} F_{ap}(\{x_i\}), \qquad F_{ap}(\{x_i\}) = \sum_{i=0}^{M} \omega_{i+1/2}^3 h_{i+1/2}^3$$

Algorithm (direct minimization)

For $k = 1, \ldots, K_{max}$ do

1. For the given grid $\{x_i^k\}$ compute values $\omega_{i+1/2}^k$, $i = 0, \ldots, M$.
2. Perform one Gauss-Seidel sweep

$$\min_{x_i^{k+1}} \left\{ \left[\hat{\omega}_{i+1/2}^{k+1} (x_{i+1}^k - x_i^{k+1}) \right]^3 + \left[\hat{\omega}_{i-1/2}^{k+1} (x_i^{k+1} - x_{i-1}^{k+1}) \right]^3 \right\},$$

   where $i = 1, \ldots, M$, R_h is the interpolation operator from grid $\{x_i^k\}$ to grid $\{x_i^{k+1}\}$, and $\hat{\omega}^{h,k+1} = R_h(\omega^{h,k})$.

3. Stop iterations if $\max_i |x_i^k - x_i^{k+1}| \leq TOL$.

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Approximate error functional (4/7)

$$\sum_{i=0}^{M} \omega_{i+1/2}^{k} h_{i+1/2}^{k} = \sum_{i=0}^{M} \hat{\omega}_{i+1/2}^{k+1} h_{i+1/2}^{k+1}.$$

Algorithm (direct minimization)

For $k = 1, \ldots, K_{max}$ do

1. For the given grid $\{x_i^k\}$ compute values $\omega_{i+1/2}^k$, $i = 0, \ldots, M$.
2. Perform one Gauss-Seidel sweep

$$\min_{x_i^{k+1}} \left\{ \left[\hat{\omega}_{i+1/2}^{k+1} \left(x_{i+1}^{k} - x_i^{k+1} \right) \right]^3 + \left[\hat{\omega}_{i-1/2}^{k+1} \left(x_i^{k+1} - x_{i-1}^{k+1} \right) \right]^3 \right\},$$

where $i = 1, \ldots, M$, R_h is the interpolation operator from grid $\{x_i^k\}$ to grid $\{x_i^{k+1}\}$, and $\hat{\omega}^{h,k+1} = R_h(\omega^{h,k})$.

3. Stop iterations if $\max_i |x_i^k - x_i^{k+1}| \leq TOL$.

Approximate error functional (5/7)

Let us consider a test function $f(x)$ given by

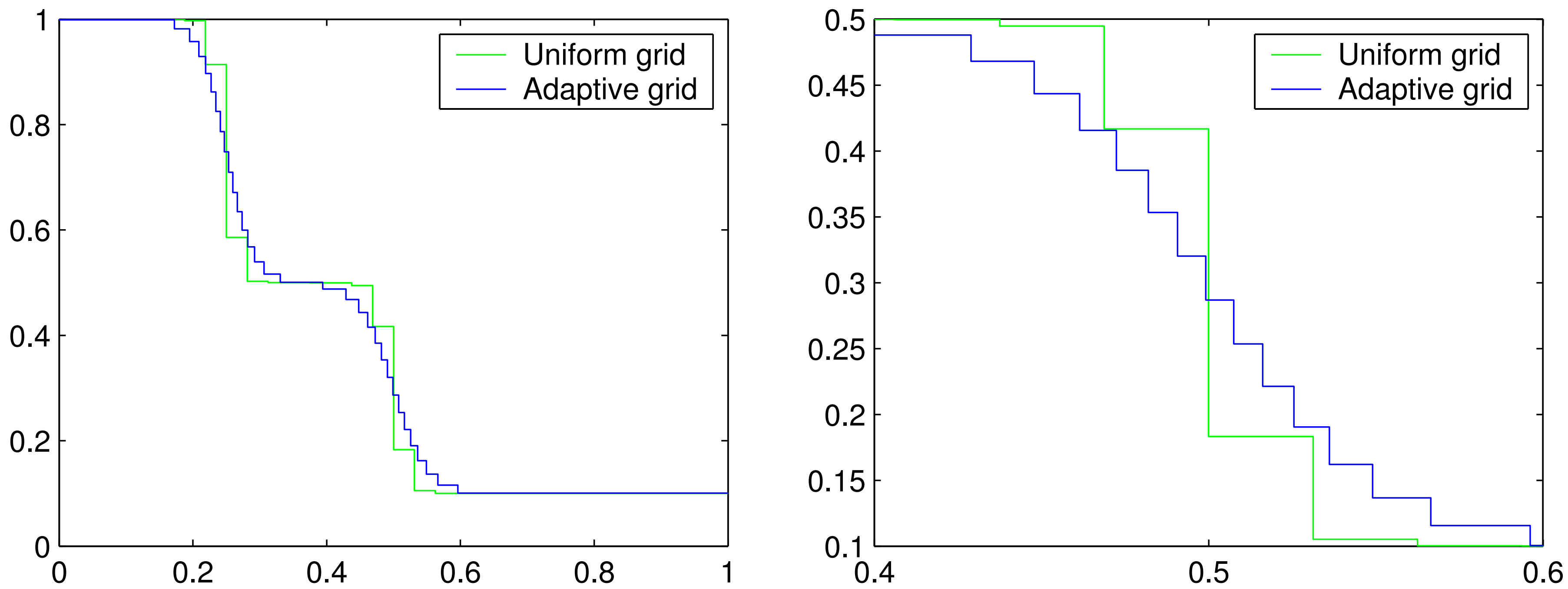
$$f(x) = 1 - \frac{9\,r_1 + 5\,r_1^5}{10(r_1 + r_1^5 + r_2)}, \quad r_1 = \exp\frac{1/2 - x}{20\varepsilon}, \quad r_2 = \exp\frac{3/8 - x}{2\varepsilon},$$

with $\varepsilon = 0.005$. Let

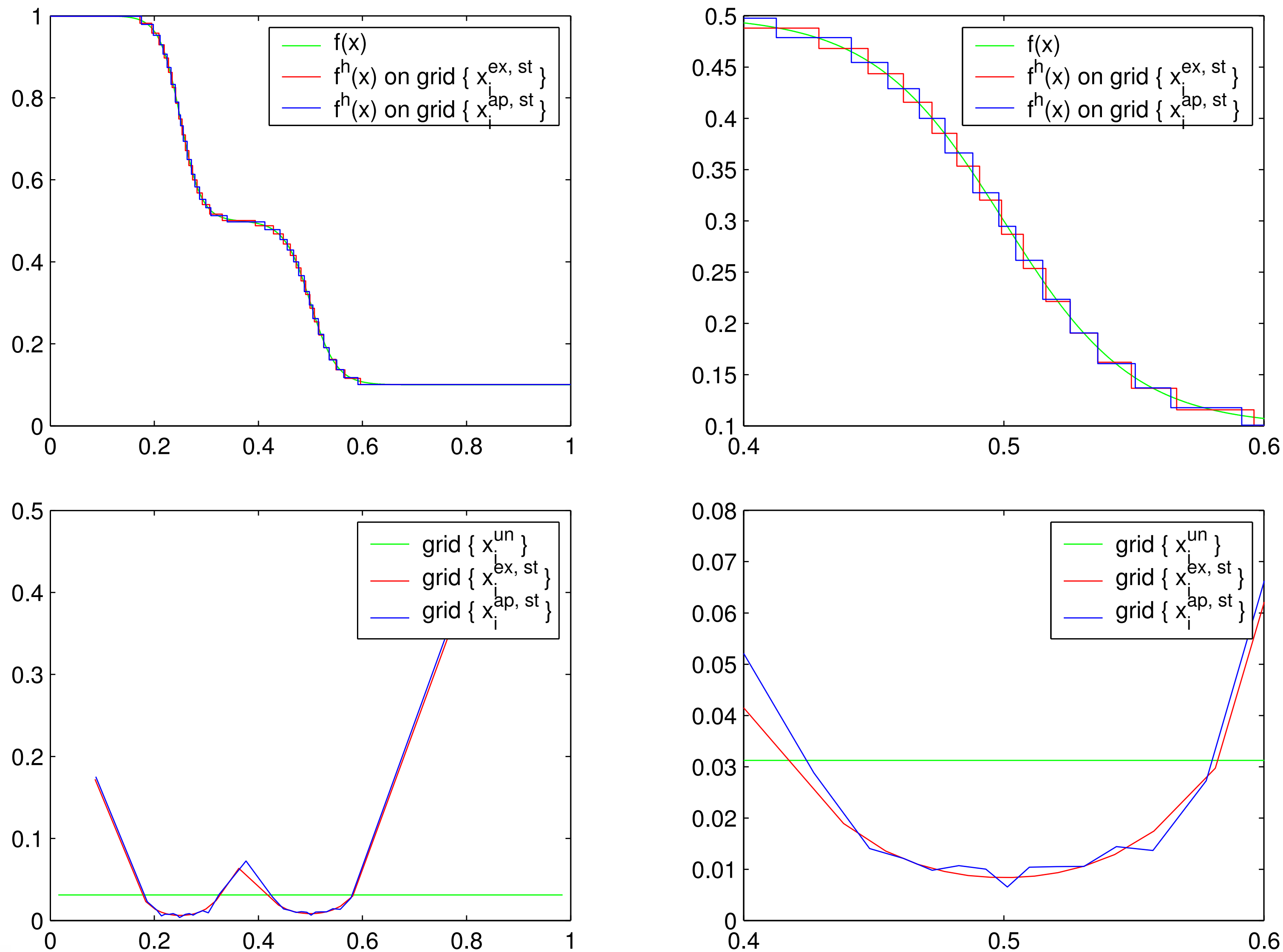
$$E(\{x_i\}) = \sqrt{F_{ex}(\{x_i\})}$$

M	$E(\{x_i^{un}\})$	$E(\{x_i^{ex,st}\})$	$E(\{x_i^{ap,st}\})$	$E(\{x_i^{ap,eq}\})$
16	2.99e-2	1.01e-2	1.19e-2	1.19e-2
32	1.59e-2	4.99e-3	5.22e-3	5.22e-3
64	7.99e-3	2.48e-3	2.51e-3	2.51e-3
128	4.00e-3	1.24e-3	1.24e-3	1.24e-3

Approximate error functional (6/7)



Approximate error functional (7/7)



Grid smoothing (1/4)

Let the mesh steps satisfy the following condition:

$$\frac{\alpha}{\alpha+1} \le \frac{h_{i-1/2}}{h_{i+1/2}} \le \frac{\alpha+1}{\alpha}, \qquad i = 1, \ldots, M.$$

Lemma. Let $\omega_{i+1/2}$, $i = 0, \ldots, M$, be given values of a monitor function. The values $\tilde{\omega}_{i+1/2}$, $i = 0, \ldots, M$, of a smoothed monitor function satisfying

$$\frac{\alpha}{\alpha+1} \le \frac{\tilde{\omega}_{i+1/2}}{\tilde{\omega}_{i-1/2}} \le \frac{\alpha+1}{\alpha}, \qquad i = 1, \ldots, M,$$

can be computed by solving the system of $M+1$ linear equations:

$$\tilde{\omega}_{i+1/2} - \alpha(\alpha+1)(\tilde{\omega}_{i+3/2} - 2\tilde{\omega}_{i+1/2} + \tilde{\omega}_{i-1/2}) = \omega_{i+1/2},$$

where $\tilde{\omega}_{-1/2} = \omega_{1/2}$ and $\tilde{\omega}_{M+3/2} = \omega_{M+1/2}$.

Grid smoothing (1/4)

Let the mesh steps satisfy the following condition:

$$\frac{\alpha}{\alpha+1} \le \frac{\omega_{i+1/2}}{\omega_{i-1/2}} \le \frac{\alpha+1}{\alpha}, \qquad i = 1, \ldots, M.$$

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$$\frac{\alpha}{\alpha+1} \le \frac{\tilde{\omega}_{i+1/2}}{\tilde{\omega}_{i-1/2}} \le \frac{\alpha+1}{\alpha}, \qquad i = 1, \ldots, M,$$

can be computed by solving the system of $M+1$ linear equations:

$$\tilde{\omega}_{i+1/2} - \alpha(\alpha+1)(\tilde{\omega}_{i+3/2} - 2\tilde{\omega}_{i+1/2} + \tilde{\omega}_{i-1/2}) = \omega_{i+1/2},$$

where $\tilde{\omega}_{-1/2} = \omega_{1/2}$ and $\tilde{\omega}_{M+3/2} = \omega_{M+1/2}$.

Grid smoothing (2/4)

Algorithm (direct minimization with smoothing)

For $k = 1, \ldots, K_{max}$ do

1. For the given grid $\{x_i^k\}$ compute values $\omega_{i+1/2}^k$, $i = 0, \ldots, M$.
2. Compute the smoothed values $\tilde{\omega}_{i+1/2}^k$, $i = 0, \ldots, M$, by solving the tridiagonal system.
3. Perform one Gauss-Seidel sweep

$$\min_{x_i^{k+1}} \left\{ \left[\hat{\omega}_{i+1/2}^{k+1} \left(x_{i+1}^k - x_i^{k+1} \right) \right]^3 + \left[\hat{\omega}_{i-1/2}^{k+1} \left(x_i^{k+1} - x_{i-1}^{k+1} \right) \right]^3 \right\},$$

   where $i = 1, \ldots, M$, R_h is the interpolation operator from grid $\{x_i^k\}$ to grid $\{x_i^{k+1}\}$, and $\hat{\omega}^{h,k+1} = R_h(\tilde{\omega}^{h,k})$.

4. Stop iterations if $\max_i |x_i^k - x_i^{k+1}| \leq TOL$ where TOL is the user given tolerance.

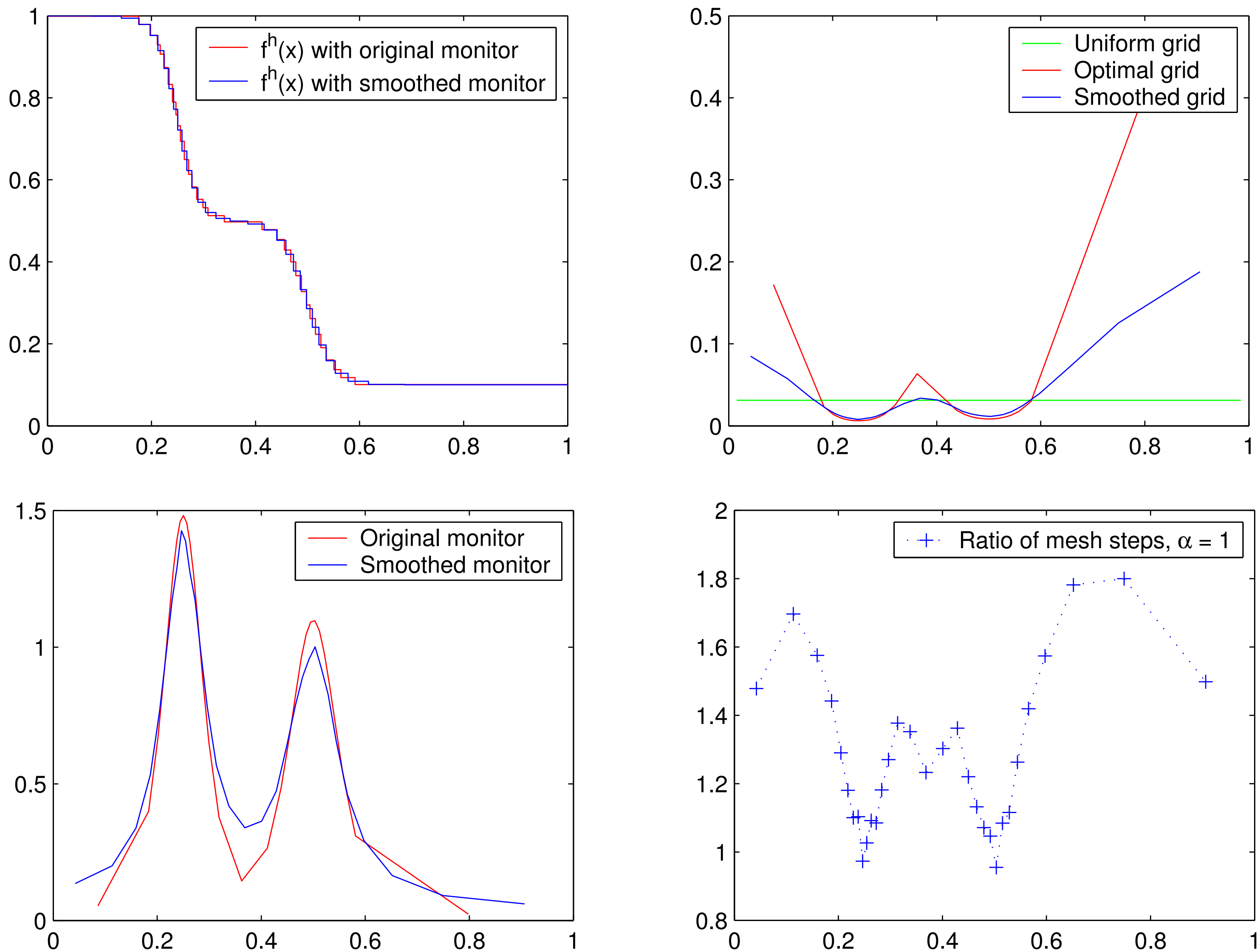
Grid smoothing (3/4)

Recall that

$$E(\{x_i\}) = \sqrt{F_{ex}(\{x_i\})}.$$

M	$E(\{x_i^{ap,st}\})$	$E(\{\tilde{x}_i^{ap,st,sm}\})$	$E(\{\tilde{x}_i^{ap,st,sm}\})$
16	1.19e-2	1.62e-2	1.68e-2
32	5.22e-3	6.01e-3	6.50e-3
64	2.51e-3	2.68e-3	2.66e-3
128	1.24e-3	1.28e-3	1.25e-3

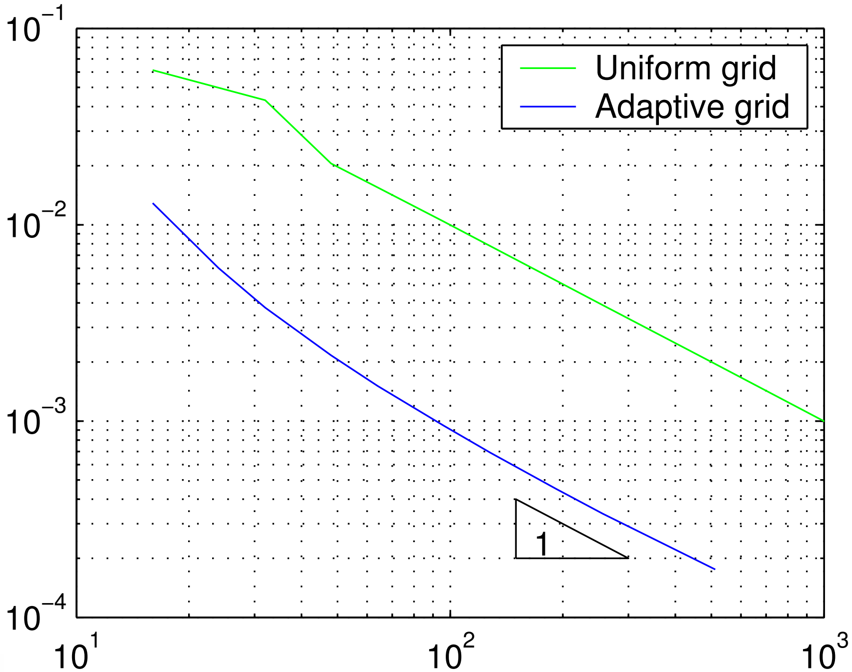
Grid smoothing (4/4)

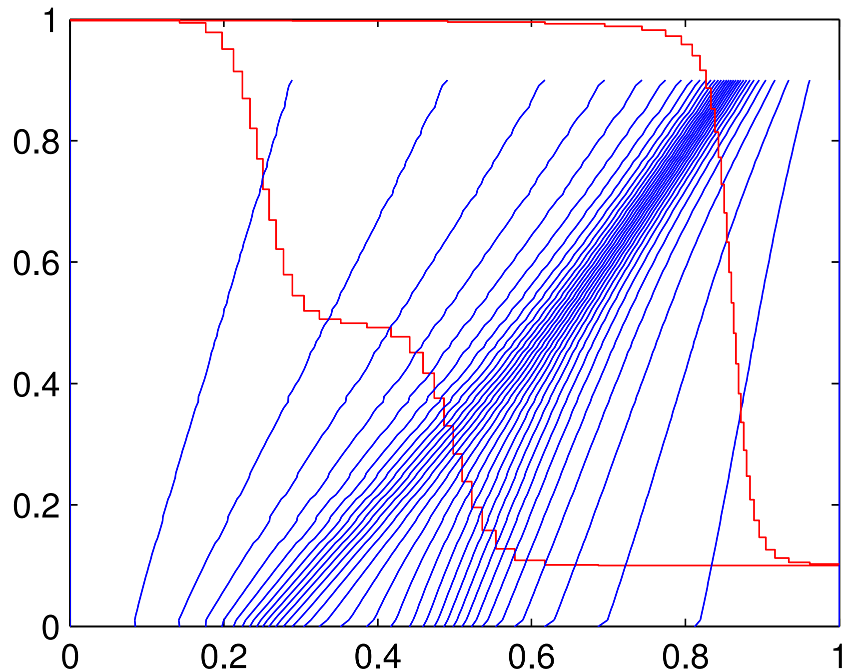


Burgers equation (1/2)

Let $T = 0.9$, $\varepsilon = 0.005$ and

$$E(\{x_i^N\}) = \left[\sum_{i=0}^{M} \int_{x_i^N}^{x_{i+1}^N} (u(x,\,T) - \bar{u}_{i+1/2}^N)^2 \mathrm{d}x \right]^{1/2}.$$

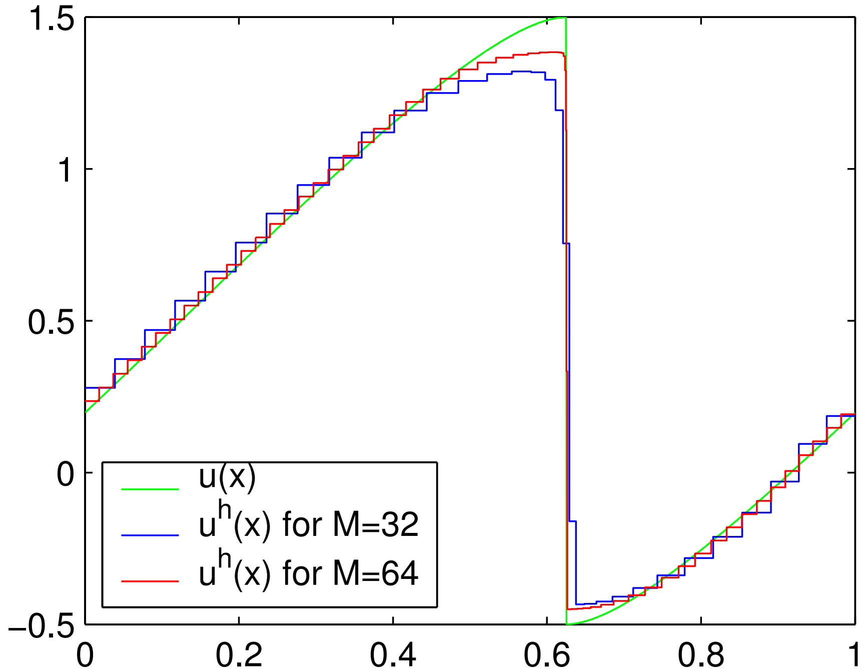


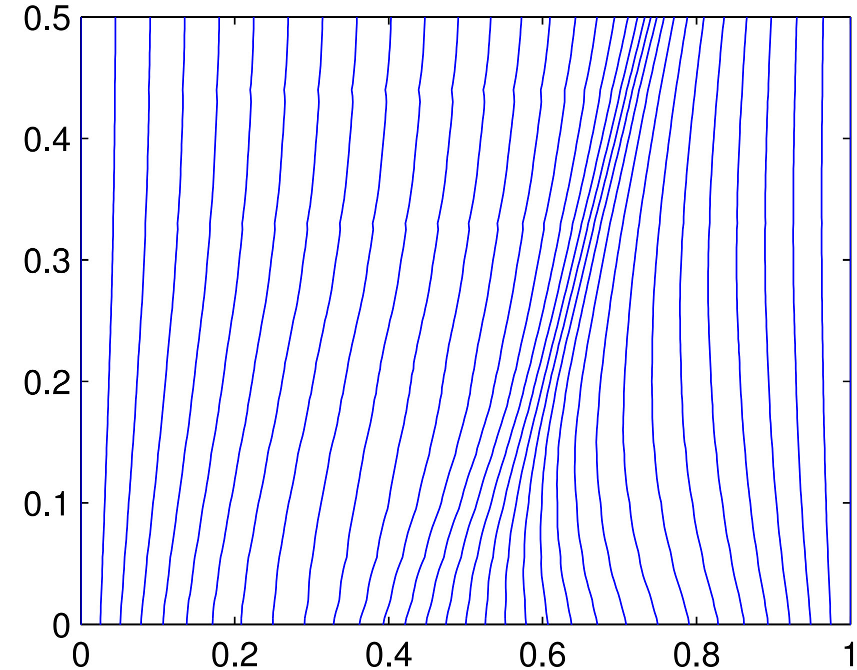


Burgers equation (2/2)

Let $T = 0.5$, $\varepsilon = 0$ and the initial condition be the periodic function

$$u_0(x) = 0.5 + \sin(2\pi x).$$





Conclusion

- The error introduced by the numerical scheme can be ignored even for lower order time integration schemes.
- The error introduced by the numerical interpolation can be ignored when the interpolation operator is more accurate than the discretization.
- Necessity of a grid smoothing has been observed in many numerical experiments.
- The algorithms have shown a robust behavior for 1D Burgers equation.